Utility Case Study
Utility-Derived Discount Rates

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In this white paper we will derive market discount rates (i.e. required rates of return) for four investments based on a generic investor utility function. We will assume that investor wealth at time zero is invested in one of four possible investments and is invested for a one year term. We will use the investor utility function to price each of these four investments and calculate the expected rates of return (ex-ante discount rates).

Risk Aversion : Investors are generally thought to be risk-averse. Risk aversion is a psychological and economic principle where individuals prefer a sure outcome (like a guaranteed payment) over a gamble even if that gamble offers a potential for greater gains. Risk-averse individuals are more comfortable with a predictable outcome than with a gamble where the final result is uncertain.

Building Our Model

We will assume that our investor's wealth is reinvested at the beginning of each year. We will define the variable W_t to be total investor wealth at time t. Our investor's wealth in dollars at time zero is...

$$W_0 = \text{Beginning wealth} = \$95,000 \tag{1}$$

We will assume that there are two possible states of the world (STOW) at the end of each year. The possible states of the world and attendant probabilities are...

Table 1: Possible States-of-the-World

STOW	Description	Probability
M_U	Market goes up	$P[M_U] = 0.50$
M_D	Market goes down	$P[M_D] = 0.50$

We will define the variable W_U to be the end of year investment payoff given that the market goes up and the variable W_D to be the end of year investment payoff given that the market goes down. We will assume that the investor has the following four investment options...

Table 2: Investment Options

Investment		One Year Payoff Amounts				
Symbol	Description	W_U	W_D	Dispersion	Expected	Notes
T	Treasury Bill	100,000	100,000	0	100,000	Investment has zero risk
A	Investment A	110,000	90,000	10,000	100,000	Riskier than T, less risky than B and C
B	Investment B	125,000	75,000	25,000	100,000	Riskier than T and A, less risky than C
C	Investment C	140,000	60,000	40,000	100,000	Investment has the most risk

In mathematical terms, we will **define risk to be the dispersion of possible results** (i.e. volatility). Note that in the table above, as dispersion gets larger, expected utililty gets smaller, and therefore the investor requires an incentive to invest in risky investment options A, B and C as compared to the risk-free investment option T.

Using Tables 1 and 2 above, the equation for expected investment payoff is...

$$\mathbb{E}\left[\text{Investment payoff}\right] = W_U \times P[M_U] + W_D \times P[M_D] \tag{2}$$

Using Equations (1) and (2) above, the equation for expected investment annual return is...

$$\mathbb{E}\left[\text{Investment annual return}\right] = \mathbb{E}\left[\text{Investment payoff}\right] \div W_0 - 1 \tag{3}$$

Using Equations (1) and (3) above and Investment T in Table 2 above, the annual risk-free interest rate is...

$$\text{Risk-free interest rate} = 100,000 \div 95,000 - 1 = 0.0526 \tag{4}$$

We will define the function $U(W)$ to be the investor's total utility of wealth. We will assume that our investor has the following log utility equation... [1] [2]

$$U(W) = a + b\,\text{Ln}\left\{c\,W\right\} \text{ ...where... } a = 10 \text{ ...and... } b = 2 \text{ ...and... } c = 0.001 \tag{5}$$

Using Equation (5) above and Tables 1 and 2 above, the equation for expected utility at the end of each investment year given the investor's chosen investment option at the beginning of the year is... [2]

$$\mathbb{E}\left[U(W)\right] = \left[a + b\,\text{Ln}\left\{c\,W_U\right\}\right] \times P[M_U] + \left[a + b\,\text{Ln}\left\{c\,W_D\right\}\right] \times P[M_D] \tag{6}$$

Using Equation (6) above and Tables 1 and 2 above, expected utility for our investment options are...

Table 3: Expected Utility

Investment		Expected Utility		
Symbol	Description	Market Up	Market Down	Expected
T	Treasury Bill	19.210	19.210	**19.210**
A	Investment A	19.401	19.000	**19.200**
B	Investment B	19.657	18.635	**19.146**
C	Investment C	19.883	18.189	**19.036**

Risky Investment Incentive

We will define the variable X to be the investment incentives applicable to each investment option in Table 2 above where the value of the incentive is such that the investor is indifferent between the risky investment and the risk-free investment. The idea is to increase the payoff on risky investments A, B and C such that expected utility from those investments plus the incentive equals expected utility from the risk free Investment T.

Using Equations (5) and (6) above, the value of the incentive applicable to the risky investment is the solution to the following equation with respect to the independent variable X...

$$\mathbb{E}\left[U(T)\right] = \left[a + b\,\text{Ln}\left\{c\left(W_U + X\right)\right\}\right] \times P[M_U] + \left[a + b\,\text{Ln}\left\{c\left(W_D + X\right)\right\}\right] \times P[M_D]$$

$$\mathbb{E}\left[U(T)\right] = a + b\,\text{Ln}\left\{c\left(W_U + X\right)\right\} P[M_U] + b\,\text{Ln}\left\{c\left(W_D + X\right)\right\} P[M_D]$$

$$\mathbb{E}\left[U(T)\right] = a + b\left[\text{Ln}\left\{c\left(W_U + X\right)\right\} P[M_U] + \text{Ln}\left\{c\left(W_D + X\right)\right\} P[M_D]\right]$$

$$\left(\mathbb{E}\left[U(T)\right] - a\right) \Big/ b = \text{Ln}\left\{c\left(W_U + X\right)\right\} P[M_U] + \text{Ln}\left\{c\left(W_D + X\right)\right\} P[M_D] \tag{7}$$

We will use the Newton-Raphson method to solve Equation (7) above. We will define the function $F(X)$ to be the left side of Equation (7) above where X is the unknown **actual value** of the investive...

$$F(X) = \left(\mathbb{E}\Big[U(T)\Big] - a\right) \Big/ b \tag{8}$$

We will define the function $F(\hat{X})$ to the solution to Equation (7) above where the variable $\hat{X}$ is our **guess value** of the incentive...

$$F(\hat{X}) = \text{Ln}\left\{c\left(W_U + \hat{X}\right)\right\} P[M_U] + \text{Ln}\left\{c\left(W_D + \hat{X}\right)\right\} P[M_D] \tag{9}$$

The equation for the derivative of Equation (9) above with respect to the incentive variable $\hat{X}$ is...

$$\frac{\delta F\hat{X})}{\delta \hat{X}} = c\left[P[M_U] \Big/ \text{Ln}\left\{c\left(W_U + X\right)\right\} + P[M_D] \Big/ \text{Ln}\left\{c\left(W_D + X\right)\right\}\right] \tag{10}$$

The Answers To Our Case Study

Rather than add the incentive value to risky investment state payoffs, we will subtract the present value of the incentive from beginning wealth such that the price of each risky investment is beginning wealth minus the present value of the incentive using the risk-free discount rate in Equation (4) above.

Using the Newton-Raphson method for solving non-linear equations (uses Equation (8), (9) and (10) above), the incentive future value (FV) and present value (PV) for our three risky-investments are... [3]

Table 4: Risky Investment A

Iteration	x + error	=	x hat	F(x)	F(x hat)	dF(x hat)	WU	WD
1	397.52		5,000.00	4.6052	4.6494	0.0000	110,000	90,000
2	498.70		397.52	4.6052	4.6042	0.000010	110,000	90,000
3	498.76		498.70	4.6052	4.6052	0.000010	110,000	90,000
4	498.76		498.76	4.6052	4.6052	0.000010	110,000	90,000
5	498.76		498.76	4.6052	4.6052	0.000010	110,000	90,000

Investment A incentive:

$$FV = 498.76 \text{ ...and... } PV = 498.76 \times \frac{1}{1.0526} = 473.82 \text{ ...and... Price} = 95,000 - 473.82 = 94,526.18 \tag{11}$$

Table 5: Risky Investment B

Iteration	x + error	=	x hat	F(x)	F(x hat)	dF(x hat)	WU	WD
1	3,057.64		5,000.00	4.6052	4.6248	0.000010	125,000	75,000
2	3,077.64		3,057.64	4.6052	4.6050	0.000010	125,000	75,000
3	3,077.64		3,077.64	4.6052	4.6052	0.000010	125,000	75,000
4	3,077.64		3,077.64	4.6052	4.6052	0.000010	125,000	75,000
5	3,077.64		3,077.64	4.6052	4.6052	0.000010	125,000	75,000

Investment B incentive:

$$FV = 3.077.64 \text{ ...and... } PV = 3.077.64 \times \frac{1}{1.0526} = 2,923.76 \text{ ...and... Price} = 95,000 - 2,923.76 = 92,076.24 \tag{12}$$

Table 6: Risky Investment C

Iteration	x + error	=	x hat	F(x)	F(x hat)	dF(x hat)	WU	WD
1	7,657.82		5,000.00	4.6052	4.5756	0.000011	140,000	60,000
2	7,703.28		7,657.82	4.6052	4.6047	0.000011	140,000	60,000
3	7,703.30		7,703.28	4.6052	4.6052	0.000011	140,000	60,000
4	7,703.30		7,703.30	4.6052	4.6052	0.000011	140,000	60,000
5	7,703.30		7,703.30	4.6052	4.6052	0.000011	140,000	60,000

Investment C incentive:

$$FV = 7,703.30 \text{ ...and... } PV = 7,703.30 \times \frac{1}{1.0526} = 7,318.13 \text{ ...and... Price} = 95,000 - 7,318.13 = 87,681.87 \quad (13)$$

Using the results of Tables 4, 5 and (6) above, the utility-derived market discount rates for our four investments are...

Table 7: Utility-Derived Discount Rates

Symbol	Price	Expected Payoff	Discount Rate
T	95,000.00	100,000.00	$100,000.000 \div 95,000.00 - 1 = 0.0526$
A	94,526.18	100,000.00	$100,000.000 \div 94,526.18 - 1 = 0.0579$
B	92,076.24	100,000.00	$100,000.000 \div 92,076.24 - 1 = 0.0861$
C	87,681.87	100,000.00	$100,000.000 \div 87,681.87 - 1 = 0.1405$

References

[1] Gary Schurman, *Introduction To Utility Funtions,* October, 2023.

[2] Gary Schurman, *The Logrithmic Utility Function,* October, 2023.

[3] Gary Schurman, *Newton-Raphson Method For Solving Nonlinear Equations,* October, 2009.